Math 2144 Final **Spring 2024** **NAME:** ______________________

CRN: ______________________

INSTRUCTIONS: This exam is a **closed book exam**. You may **not** use your text, homework, or other aids except for two 3×5-inch notecards. You may use an approved calculator to perform operations on real numbers, evaluate functions at specific values, and look at graphs and/or tables.

A TI-89, TI-Nspire, or a calculator with a computer algebra system, any technology with wireless or Internet capability (i.e., laptops, tablets, smart phones or watches), a QWERTY keyboard, or a camera are **not allowed**. Unless otherwise stated, you must **show all of your work**, including all steps needed to solve each problem and explain your reasoning to earn full credit. This means that **correct answers using incorrect reasoning may not receive any credit**. This comprehensive exam assesses your understanding of material covered throughout the entire course.

Please write legibly. Your exam will be scanned and graded using an online grading system. The grader will not be able to interpret your work if you do not write clearly and dark enough.

Turn off all noise-making devices and all devices with an internet connection and put them away. Put away all headphones, earbuds, etc.

You have **1 hour** and **50 minutes** to work starting from the signal to begin.

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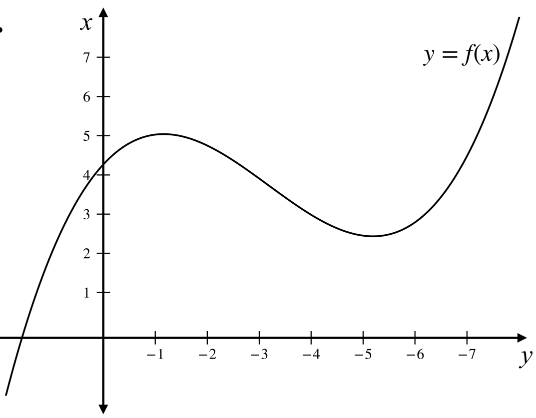
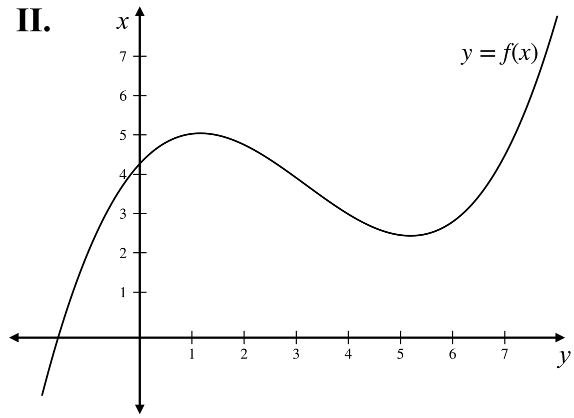
1. (*2 points each*) Answer the following multiple choice questions. No justification or explanation is required.

   (i) For which of the following graphs of $y = f(x)$ is $f'(3) < 0$? Select all that apply. (*Hint: Pay attention to the labeling and the scales of the axes.*)

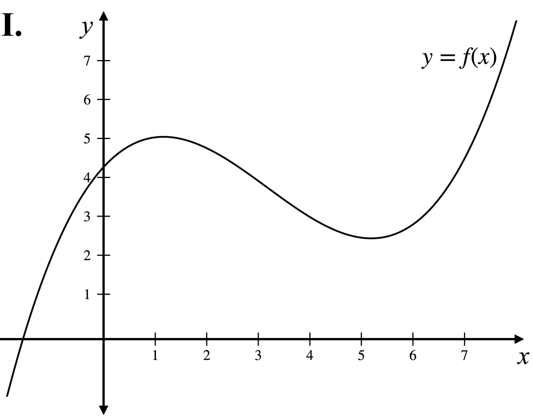
I.



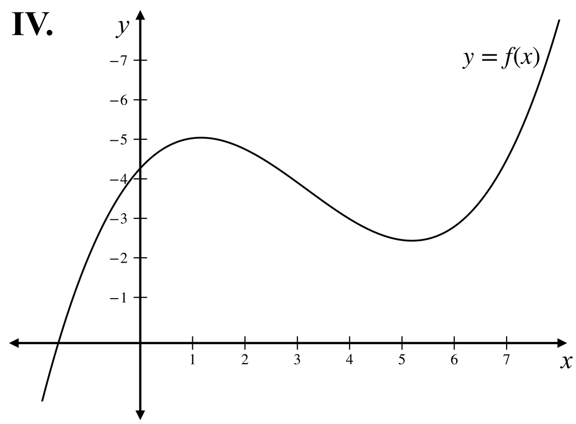
II.



III.



IV.



A. I only

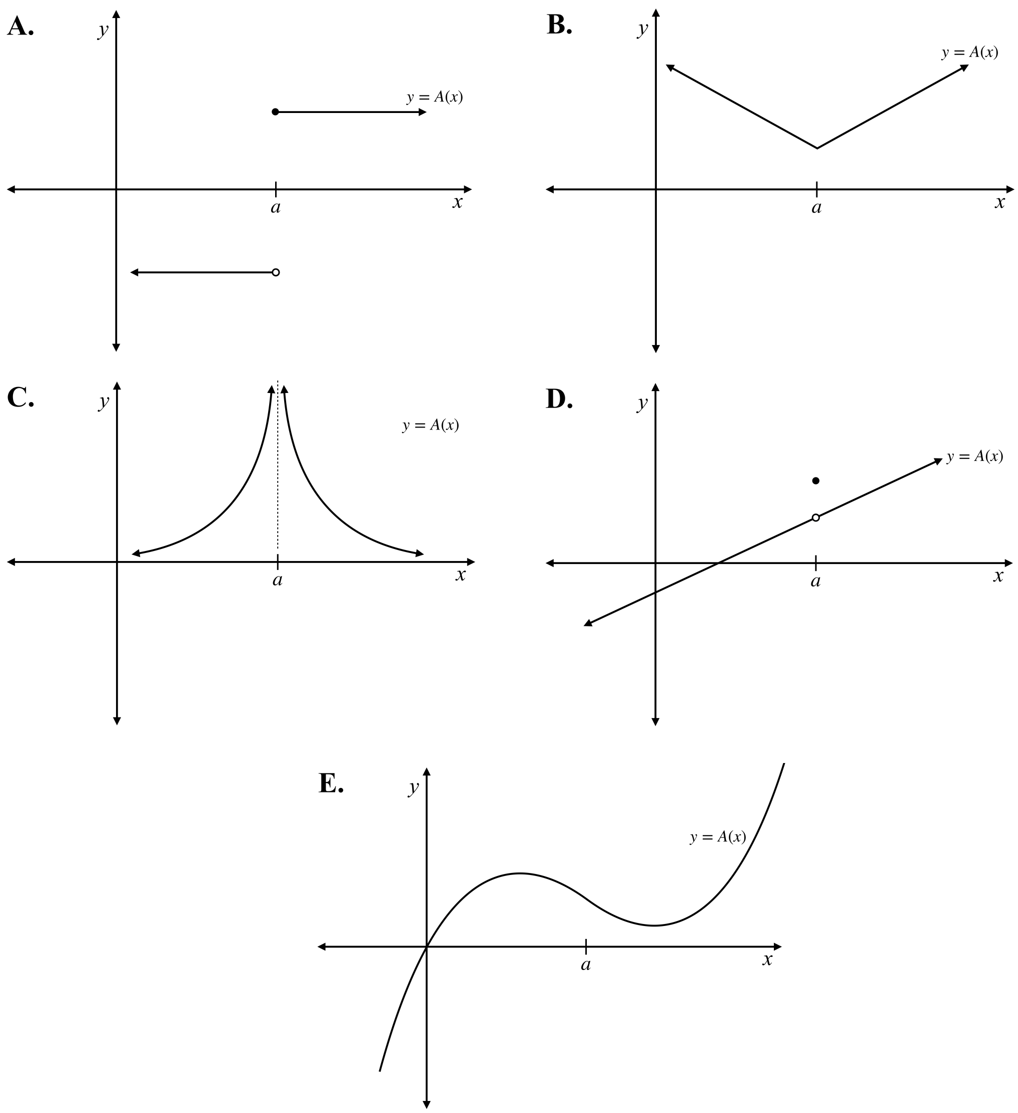
B. II only

C. II and III only

D. III only

E. I and IV only

(ii) Let $A(x) = \displaystyle\int_{-1}^{x} f(t)\,dt$. Suppose $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but are not equal. If $\lim\limits_{x\to a} A(x)$ exists, which of the following could be the graph of $y = A(x)$?



A. Graph A

B. Graph B

C. Graph C

D. Graph D

E. Graph E

(iii) Oil leaks out of a tank at a rate of $r = f(t)$ gallons per minute, where t is measured in minutes. Which of the following expressions represents the **exact** amount of oil that leaked out of the tank from 15 to 45 minutes after oil started leaking?

I. $f(45) - f(15)$

II. $\displaystyle\int_{15}^{45} f(t)\, dt$

III. $\displaystyle\sum_{k=1}^{60} f\left(15 + \frac{k}{2}\right) \cdot \frac{1}{2}$

IV. $\displaystyle\lim_{N\to\infty} \sum_{k=1}^{N} f\left(15 + \frac{30k}{N}\right) \cdot \frac{30}{N}$

a. I only

b. I and II only

c. II and IV only

d. I, II, and IV only

e. I, II, III, and IV

(iv) $\displaystyle\frac{d}{dx} \int_{2}^{h(x)} f(t)\, dt =$

a. $f(h(x)) \cdot h'(x)$

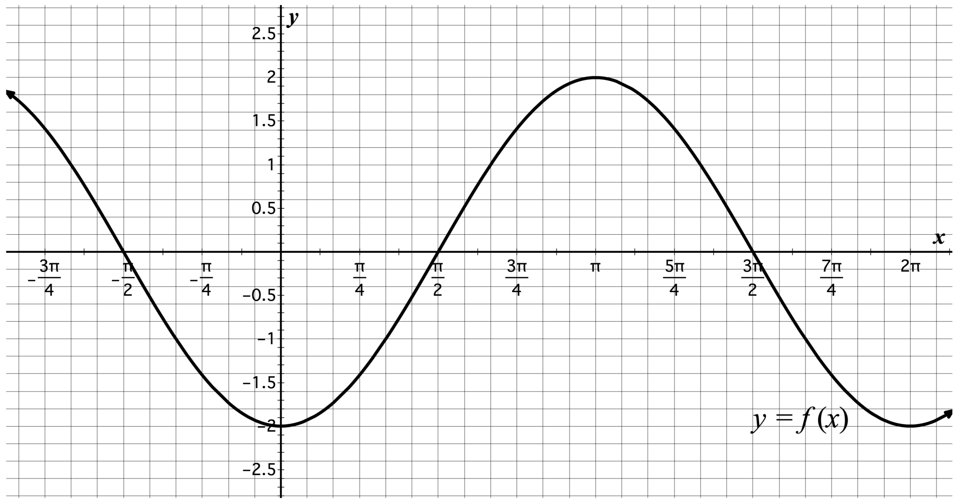
b. $f'(h(x)) \cdot h'(x)$

c. $f(h(x)) - f(2)$

d. $f(h(x))$

e. $f'(h(x))$

(v) The graph of the function f is below. The following inequalities compare the values of the quantities $f'\left(\frac{\pi}{2}\right)$, $f''(\pi)$, $\dfrac{f(\pi) - f\left(\frac{\pi}{2}\right)}{\pi - \frac{\pi}{2}}$, and $\displaystyle\int_{\pi}^{2\pi} f(x)\,dx$. Which of the following string of inequalities is true?



a. $\displaystyle\int_{\pi}^{2\pi} f(x)\,dx > f''(\pi) > \frac{f(\pi) - f\left(\frac{\pi}{2}\right)}{\pi - \frac{\pi}{2}} > f'\left(\frac{\pi}{2}\right)$

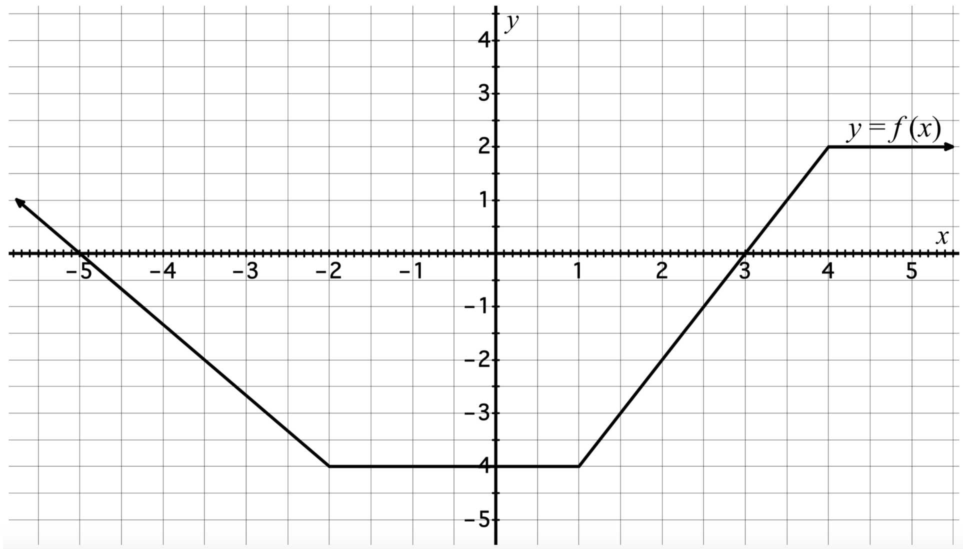
b. $\displaystyle f'\left(\frac{\pi}{2}\right) > \frac{f(\pi) - f\left(\frac{\pi}{2}\right)}{\pi - \frac{\pi}{2}} > \int_{\pi}^{2\pi} f(x)\,dx > f''(\pi)$

c. $\displaystyle \frac{f(\pi) - f\left(\frac{\pi}{2}\right)}{\pi - \frac{\pi}{2}} > f'\left(\frac{\pi}{2}\right) > \int_{\pi}^{2\pi} f(x)\,dx > f''(\pi)$

d. $\displaystyle f''(\pi) > \int_{\pi}^{2\pi} f(x)\,dx > \frac{f(\pi) - f\left(\frac{\pi}{2}\right)}{\pi - \frac{\pi}{2}} > f'\left(\frac{\pi}{2}\right)$

e. $\displaystyle f'\left(\frac{\pi}{2}\right) > f''(\pi) > \int_{\pi}^{2\pi} f(x)\,dx > \frac{f(\pi) - f\left(\frac{\pi}{2}\right)}{\pi - \frac{\pi}{2}}$

(vi) Let the function $A(x) = \int_0^x f(t)\,dt$, where f is the function graphed below. Which of the following are the correct values for $A(-2)$ and $A''(3)$?



a. $A(-2) = -8$ and $A''(3) = 0$

b. $A(-2) = 8$ and $A''(3) = 0$

c. $A(-2) = -8$ and $A''(3) = 2$

d. $A(-2) = -8$ and $A''(3) = 3$

e. $A(-2) = 8$ and $A''(3) = 2$

(vii) The functions f, g, and h are defined as follows:

$$f(x) = \frac{x^2 - 1}{x - 1} \qquad g(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases} \qquad h(x) = x + 1$$

Which of the following is true?

I. $\lim_{x \to 1} g(x) = g(1)$

II. $\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \lim_{x \to 1} h(x)$

III. $f(1) = g(1) = h(1)$

a. I only
b. I and II
c. II only
d. II and III
e. I, II, and III

(viii) For which of the functions $y = f(x)$ would it be appropriate to apply L'Hôpital's Rule to evaluate the following limit?

$$\lim_{x\to 1} \frac{f(x)}{\sqrt{\ln(x)}}$$

a. $f(x) = \sin(x)$

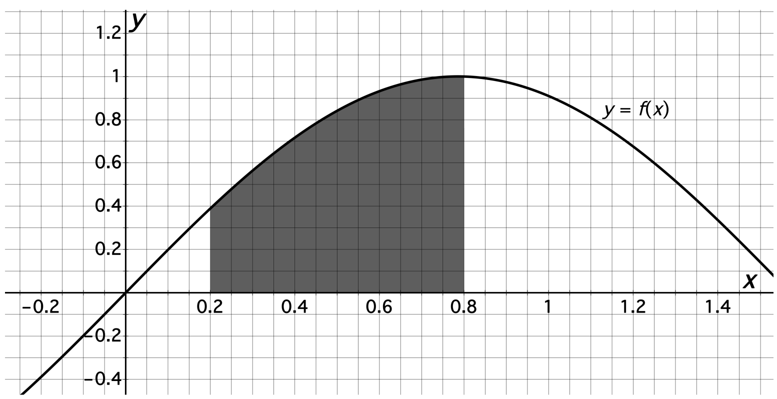
b. $f(x) = e^x$

c. $f(x) = x^2 - 1$

d. $f(x) = \sec(x - 1)$

e. $f(x) = \sin^{-1}(x)$

(ix) The graph of the function $f(x) = \sin(2x)$ is shown below.



The numerical value of the expression $\int_{0.2}^{0.8} \sin(2x)\,dx$ is equal to the area of the shaded region in the graph above. The value of which of the following expressions is also equal to the area of the shaded region?

I. $\dfrac{1}{2}\displaystyle\int_{0.4}^{1.6} \sin(\theta)\,d\theta$

II. $\Delta x \displaystyle\sum_{k=1}^{N} \sin\left(2(0.2 + k\Delta x)\right)$

III. $\dfrac{1}{2}\left(-\cos(1.6) + \cos(0.4)\right)$

a. I only
b. II only
c. III only
d. I and III only
e. I, II, and III

2. (*4 points*) Evaluate the following limit or state that it does not exist ("DNE"). Use ∞ or $-\infty$ if either is appropriate. A numerical answer alone without justification (either algebra and/or quoting theorems where applicable) will receive no credit. **If you use L'Hôpital's Rule, clearly indicate when you do so**. *Write your final answer in the space provided.*

$$\lim_{x\to 1} \frac{x^4-1}{x^3-1} =$$

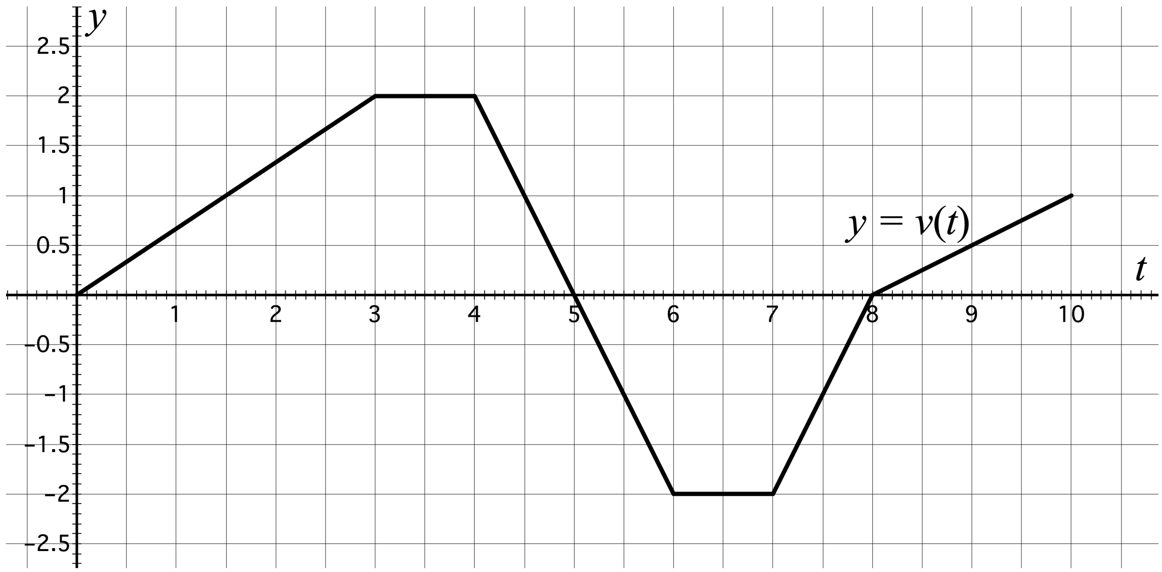
Answer: $\boxed{}$

3. (*4 points each*) Compute the following derivatives. **You do not need to simplify**.

(a) Let $y = 2^x \cdot \ln(x^2)$. Find $\dfrac{dy}{dx}$.

(b) Let $\theta = \sin^{-1}\left(\sqrt{x}\right)$. Find $\dfrac{d\theta}{dx}$ in terms of x only.

4. (*3 points each*) The graph of the function $y = v(t)$ is given below. The function v represents the vertical velocity of a car on a roller coaster (in meters per second) in terms of the number of seconds elapsed since the roller coaster started moving. At the beginning of the ride ($t = 0$), the car is 20 meters above the ground. (*Note that the graph is piecewise linear.*)



(a) How high does the car climb in the first 3 seconds of the ride?

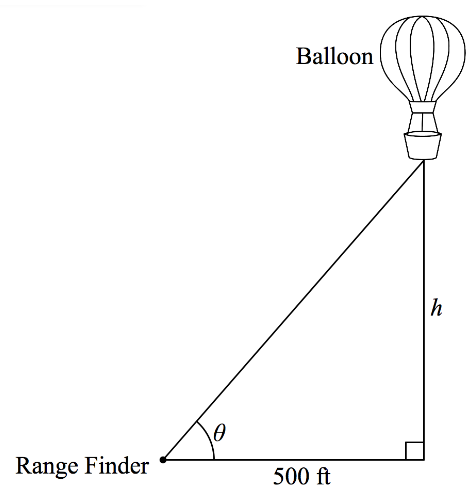
(b) What is the car's height at $t = 6$?

(c) For what value of t is the height of the car minimized?

(d) What was the car's acceleration at $t = 9$?

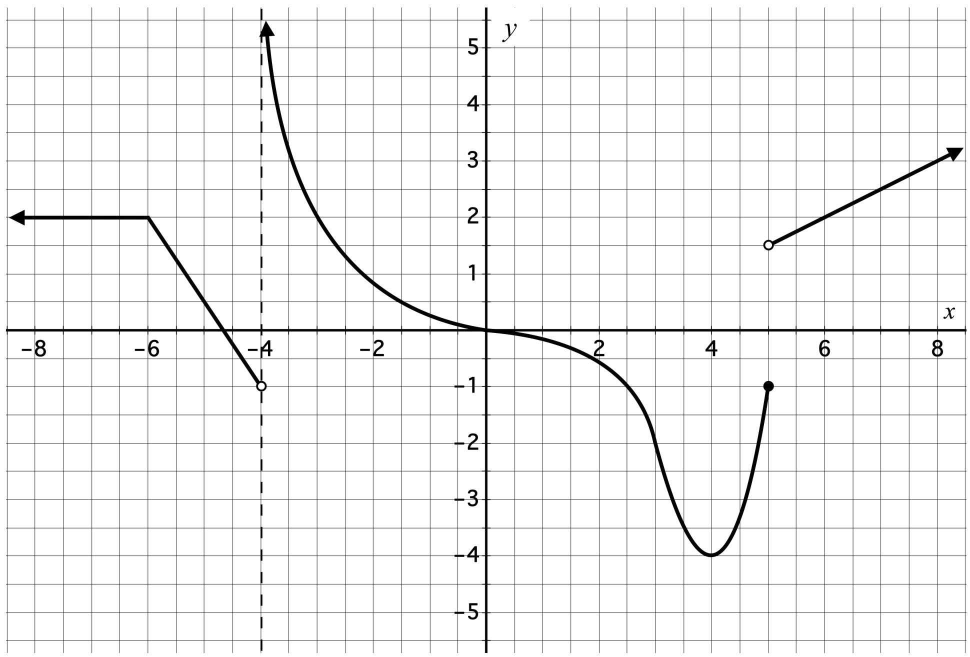
5. (*5 points*) Find $\dfrac{dy}{dx}$ for the curve $e^y - x^3 = x \cdot \cot(y^2)$.

6. (*7 points*) A hot air balloon rising straight up from a level field is tracked by a range finder 500 feet from the lift-off point (see the image below). At the moment the range finder's elevation angle is $\pi/4$ radians, the angle is increasing at a rate of 0.14 radians per minute. How fast is the balloon rising at that moment? *Express your solution accurate to three decimal places.*



Answer: $\boxed{}$ ft/min

7(a) (*9 points*) Answer the following questions based on the graph of $y = f(x)$ below. Assume that all critical points, points of discontinuity, points of inflection, and the end behavior of f can be observed from the graph below. Asymptotes are indicated by dotted lines.



Give numeric values for each of the following. Write "DNE" if the value does not exist and "∞" or "$-\infty$" as appropriate.

$\lim_{x \to -4^+} f(x) =$

$f'(8) =$

$\lim_{x \to 5^-} f(x) =$

$\int_{-8}^{-6} f(x)\,dx =$

$\left.\frac{d^2 f}{dx^2}\right|_{x=-5} =$

$\lim_{\Delta x \to 0} \frac{f(4+\Delta x) - f(4)}{\Delta x} =$

$\frac{f(7.3) - f(6)}{7.3 - 6} =$

$\lim_{x \to \infty} \frac{df}{dx} =$

$f''(0) =$

7(b) (*1 point*) Identify a value of x for which f is continuous but not differentiable.

$x =$ $\boxed{}$

8. (*4 points each*) Evaluate each of the following. Show all of your work and give exact answers. If the answer is a number, **do not round**.

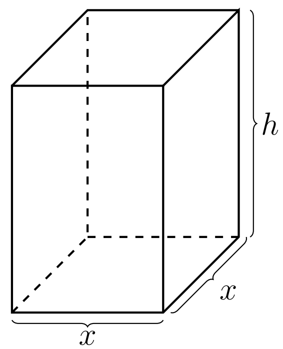
   (a) $\int_0^{\pi} (1 + \cos(x))dx =$

   (b) $\int (e^x - 4\sin(x))\, dx =$

   (c) $\frac{d}{dx}\int_1^{\ln(x)} \sqrt{2t^4 + t + 1}dt =$

   (d) $\int x^2\sqrt{5 + 2x^3}dx =$

9. (*10 points*) A box of volume 100 in^3 with square bottom and top is constructed out of two different materials. All six sides of the box will be constructed: the cost of the bottom is 4 cents per in^2 and the cost of the top and the sides is 1 cent per in^2. (*Note that the box is not necessarily a cube.*)



(a) (*2 points*) Write a formula for the **surface area** of the box as a function of a single variable.

(b) (*2 points*) Write a formula for the **cost** of manufacturing the box as a function of a single variable.

(c) (*6 points*) Compute the minimum cost of constructing the box and write your answer in the space provided.

Answer: $\boxed{}$

10. (*8 points*) The function A is given by

$$A(x) = \int_0^{x^2} \left(\sqrt{t} - 10\right)\, dt$$

where x is positive. Determine the coordinates of the point of inflection of the graph of $y = A(x)$. *Write the x and y coordinates in the spaces provided.*

x-coordinate: $\boxed{}$ y-coordinate: $\boxed{}$

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BASIC FORMULAS

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

$$\frac{d}{dx}a^x = a^x\ln(a)$$

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{du}{u} = \ln|u| + C$$

$$\int e^u\,du = e^u + C$$

$$\int a^u\,du = \frac{a^u}{\ln a} + C$$

$$\int \sin(u)\,du = -\cos(u) + C$$

$$\int \cos(u)\,du = \sin(u) + C$$

$$\int \sec^2(u)\,du = \tan(u) + C$$

$$\int \csc^2(u)\,du = -\cot(u) + C$$

$$\int \sec(u)\tan(u)\,du = \sec(u) + C$$

$$\int \csc(u)\cot(u)\,du = -\csc(u) + C$$

$$\int \tan(u)\,du = \ln|\sec u| + C$$

$$\int \cot(u)\,du = \ln|\sin u| + C$$

$$\int \sec(u)\,du = \ln|\sec u + \tan u| + C$$

$$\int \csc(u)\,du = \ln|\csc u + \cot u| + C$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int f(u(x))\cdot u'(x)\,dx = \int f(u)\,du$$

$$L_n = \Delta x\sum_{k=1}^{n} f(a+(k-1)\Delta x)$$

$$R_n = \Delta x\sum_{k=1}^{n} f(a+k\Delta x)$$

$$M_n = \Delta x\sum_{k=1}^{n} f\left(a+\frac{2k-1}{2}\Delta x\right)$$