REVIEW FOR EXAM II

This exam covers sections 4.1-4.7 and 4.9-4.10 in the book. As before this review is basically a checklist. For detailed examples and solutions you should consult your class notes, the book, and the quiz solutions posted on the course website.

BASIC FACTS ABOUT SECOND ORDER LINEAR EQUATIONS

1. Existence and uniqueness.

This chapter is concerned with equations of the following form:

$$y'' + p(t)y' + q(t)y = g(t)$$

An *initial value problem* consists of this equation together with the two requirements $y(t_0) = y_0$ and $y'(t_0) = y'_0$, where t_0, y_0, and y'_0 are given numbers. A *solution* to the initial value problem is a function $y = \phi(t)$ which satisfies the equation and the two initial conditions. As before we can ask whether a solution exists, whether it is unique, and for what values of t it is valid.

Suppose t_0 is contained in an open interval (a, b) and that $p(t)$, $q(t)$, and $g(t)$ are continuous for all $t \in (a, b)$. Then a solution exists, it is unique, and it is valid for all $t \in (a, b)$.

A typical problem concerning existence and uniqueness might give you t_0, y_0, y'_0 and an equation in the form

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = b(t)$$

and ask you for the largest interval on which the initial value problem has a unique solution. To find this out divide both sides by $a_2(t)$ to get an equation in the standard form given above. This will tell you what $p(t)$, $q(t)$, and $g(t)$ are. Locate the points at which they are discontinuous. These will chop the real number line into pieces which will be open intervals. Find the interval which contains t_0.

2. Homogeneous equations.

(a) Fundamental solutions.

Our equation is *homogeneous* if its right hand side is zero for all t, so it has the following form:

$$y'' + p(t)y' + q(t)y = 0$$

Consider an interval (a, b) on which $p(t)$ and $q(t)$ are continuous. Let y_1 and y_2 be solutions of the equation. A basic property of solutions of a homogeneous equation is the *principle of superposition*, which says that for any constants c_1 and c_2 the function $c_1y_1 + c_2y_2$ is also a solution of the homogeneous equation.

Two solutions y_1 and y_2 of our equation constitute a *fundamental set of solutions* if *every* solution of the equation on (a, b) can be written in the form $c_1y_1 + c_2y_2$ for some choice of constants c_1 and c_2.

Define the *Wronskian* of two functions f and g at t_1 to be the quantity

$$W(f,g)(t_1) = \det \begin{bmatrix} f(t_1) & g(t_1) \\ f'(t_1) & g'(t_1) \end{bmatrix} = \begin{vmatrix} f(t_1) & g(t_1) \\ f'(t_1) & g'(t_1) \end{vmatrix} =$$

$$f(t_1)g'(t_1) - g(t_1)f'(t_1).$$

The basic result about fundamental solutions is that given two solutions y_1 and y_2 of the homogeneous equation, the following statements are equivalent:

(i) y_1 and y_2 constitute a fundamental set of solutions on (a, b).

(ii) $W(y_1, y_2)(t_1) \neq 0$ for *some* point t_1 in (a, b).

(iii) $W(y_1, y_2)(t) \neq 0$ for *every* point t in (a, b).

One ingredient in this result is *Abel's Theorem*, which says that

$$W(y_1, y_2)(t) = c \exp\left(-\int p(t)\,dt\right).$$

Since the exponential function is never zero, W will never be zero unless the constant c is zero, in which case W is zero everywhere. So, if you are given two functions whose Wronskian is sometimes zero and sometimes non-zero on (a, b) you know that these two functions cannot be solutions of a second order homogeneous linear differential equation on (a, b).

(b) Initial value problems.

If you have a fundamental set $\{y_1, y_2\}$ of solutions of a homogeneous equation and an initial value problem $y(t_0) = y_0$, $y'(t_0) = y_0'$, then $y = c_1y_1 + c_2y_2$ and $y' = c_1y_1' + c_2y_2'$. Plugging in $t = t_0$, $y = y_0$, and $y' = y_0'$ gives a system of two equations in the two unknowns c_1 and c_2. Solving for c_1 and c_2 gives you the solution $y = c_1y_1 + c_2y_2$ of the initital value problem.

3. Non-homogeneous equations.

(a) General solutions.

If the right hand side of our equation

$$y'' + p(t)y' + q(t)y = g(t)$$

is non-zero for at least one point in (a, b), then our equation is *non-homogeneous*. Suppose $y_p(t)$ is *one* solution of our non-homogeneous equation. ($y_p(t)$ is called a *particular solution*.) If $z_p(t)$ is another solution of our non-homogeneous equation, then $z_p(t) - y_p(t)$ will be a solution of the homogeneous equation that we get by replacing $g(t)$ by zero.

$$y'' + p(t)y' + q(t)y = 0$$

Suppose y_1 and y_2 constitute a fundamental set of solutions for this homogeneous equation, so it has general solution $y_h = c_1 y_1 + c_2 y_2$. So $z_p - y_p = y_h = c_1 y_1 + c_2 y_2$, and hence $z_p = c_1 y_1 + c_2 y_2 + y_p$. Therefore the non-homogeneous equation has general solution

$$y = c_1 y_1 + c_2 y_2 + y_p.$$

Note that there is no constant multiplying the y_p.

(b) Initial value problems.

Given an initial value problem for a non-homogeneous equation with general solution $y = c_1 y_1 + c_2 y_2 + y_p$, compute $y' = c_1 y_1' + c_2 y_2' + y_p'$. Plug in $t = t_0$, $y = y_0$, and $y' = y_0'$, and solve for c_1 and c_2.

FINDING SOLUTIONS TO HOMOGENEOUS EQUATIONS

1. Reduction of order.

Suppose you know one non-zero solution y_1 of the homogeneous equation $y'' + py' + qy = 0$. Then you can find a second solution y_2 so that $\{y_1, y_2\}$ is a fundamental set of solutions.

Set $y_2 = vy_1$, where v is an unknown function of t which we will determine. Compute the derivatives y_2' and y_2''. These will be formulas involving the unknown functions v, v', and v'' as well as known functions that come from differentiating the known function y_1. Plug these into your differential equation and simplify. You will get an equation

$$v'' y_1 + (2y_1' + py_1)v' = 0.$$

Don't bother memorizing this formula. You will automatically get an equation of the form $F(t)v'' + G(t)v' = 0$, which you can rewrite as

$$v'' + h(t)v' = 0.$$

Let $u = v'$. Then $u' = v''$, and we get a first order equation

$$u' + h(t)u = 0$$

which is linear and is also separable. Use the methods developed previously for solving such equations to find the general solution for u. This will contain a constant c. Then integrate u to obtain v; you don't need a $+C$ when you do this integral. Multiply v times y_1 to obtain y_2. This will still contain your constant c. Set c equal to some convenient value, but make sure that doing so does not give you a function which is a constant multiple of y_1. If you have any terms in your y_2 which are constant multiples of y_1 you can discard them.

2. Constant coefficient homogeneous equations.

   Consider equations of the form $ay'' + by' + cy = 0$, where a, b, and c are constants. By plugging $y = e^{rt}$ into this equation we get $(ar^2 + br^2 + c)e^{rt} = 0$. Dividing by e^{rt} gives the *characteristic equation*

   $$ar^2 + br + c = 0.$$

   There are <u>three cases</u> depending on the nature of the roots of this quadratic equation.

   (a) Two real roots $r_1 \neq r_2$.

   Then $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ constitute a fundamental set of solutions.

   (b) One repeated real root r.

   Then $y_1 = e^{rt}$ and $y_2 = te^{rt}$ constitute a fundamental set of solutions.

(c) Two non-real roots $r_1 = \alpha + \beta\, i$, $r_2 = \alpha - \beta\, i$.

Then $y_1 = e^{\alpha t}\cos(\beta t)$ and $y_2 = e^{\alpha t}\sin(\beta t)$ constitute a fundamental set of solutions.

3. Cauchy-Euler equations

These are equations of the form $at^2 y'' + bty' + cy = 0$.

First note that THESE ARE NOT CONSTANT COEFFICIENT EQUATIONS! They have DIFFERENT SOLUTIONS from constant coefficient equations!

The philosophy behind solving these is to try a solution of the form $y = t^r$, NOT $y = e^{rt}$! Plugging $y = t^r$ into the equation and simplifying shows that r must satisfy the following characteristic equation.

$$ar^2 + (b - a)r + c = 0$$

Note that the coefficient of r is $b - a$, NOT b.

There will be three cases which depend on whether there are two real roots, one real root, or no real root. There will also be differences in the formulas depending on whether $t > 0$ or $t < 0$. (The point $t = 0$ is a singularity at which bad things can happen; we will not be considering it.)

First consider the case $t > 0$.

If the characteristic equation has two distinct real roots r_1 and r_2, then the solution is

$$y = c_1 t^{r_1} + c_2 t^{r_2}.$$

If the characteristic equation has only one real root r, then the solution is

$$y = c_1 t^r + c_2 t^r \ln t.$$

If the characteristic equation has nonreal roots $\alpha \pm \beta i$, then the solution is

$$y = c_1 t^\alpha \cos(\beta \ln t) + c_2 t^\alpha \sin(\beta \ln t).$$

For the case in which $t < 0$ replace t by $-t$ in each of the formulas above.

FINDING SOLUTIONS TO NON-HOMOGENEOUS EQUATIONS

1. Undetermined coefficients.

Suppose our equation has the form

$$ay'' + by' + cy = g(t)$$

where a, b, and c are <u>constants</u>. The general idea of undetermined coefficients is to guess a solution y_p that "looks like" $g(t)$. You will make an initial guess according to the rules given below. This guess may have to be modified later.

We first introduce some notation. We let

$p(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n,$

$P(t) = A_0 t^n + A_1 t^{n-1} + \cdots + A_{n-1} t + A_n,$

$q(t) = b_0 t^n + b_1 t^{n-1} + \cdots + b_{n-1} t + b_n$, and

$Q(t) = B_0 t^n + B_1 t^{n-1} + \cdots + B_{n-1} t + B_n.$

If $g(t) = p(t)$, then the initial guess is $y_p(t) = P(t)$.

If $g(t) = me^{\alpha t}$, then the initial guess is $y_p(t) = Ae^{\alpha t}$.

If $g(t) = k \cos \beta t$, $\ell \sin \beta t$, or $k \cos \beta t + \ell \sin \beta t$, then the initial guess is $y_p(t) = A \cos \beta t + B \sin \beta t$.

If $g(t) = p(t)e^{\alpha t}$, then the initial guess is $y_p(t) = P(t)e^{\alpha t}$.

If $g(t) = p(t) \cos \beta t$, $q(t) \sin \beta t$, or $p(t) \cos \beta t + q(t) \sin \beta t$, then the initial guess is $y_p(t) = P(t) \cos \beta t + Q(t) \sin \beta t$.

If $g(t) = ke^{\alpha t} \cos \beta t$, $\ell e^{\alpha t} \sin \beta t$, or $e^{\alpha t}(k \cos \beta t + \ell \sin \beta t)$, then the initial guess is $y_p(t) = e^{\alpha t}(A \cos \beta t + B \sin \beta t)$.

If $g(t) = e^{\alpha t} p(t) \cos \beta t$, $e^{\alpha t} q(t) \beta t$, or $e^{\alpha t}(p(t) \cos \beta t + q(t) \sin \beta t)$, then the initial guess is $y_p(t) = e^{\alpha t}(P(t) \cos \beta t + Q(t) \sin \beta t)$.

In practice, instead of using subscipted coefficients like A_0, A_1, and so on, one can just start using letters of the alphabet A, B, and so forth.

Once you make an initial guess, examine it to see whether <u>any</u> of its terms is a solution of the corresponding homogeneous equation. (Remember that a <u>term</u> is one of the things you <u>add</u> to get the guess. For example if your guess is $(At + B)e^t \sin(t) + (Ct + D)e^t \cos(t)$ then its terms are $Ate^t \sin(t)$, $Be^t \sin(t)$, $Cte^t \cos(t)$, and $De^t \cos(t)$; note that e^t, $\cos(t)$, and $\sin(t)$ are NOT terms of the guess.) If AT LEAST ONE of the terms is a solution of the homogeneous equation, then multiply the ENTIRE guess by t. If you still have some term being a solution of the homogeneous equation, then multiply the entire new guess again by t. Continue until no term is a solution of the homogeneous equation, but DO NOT go any further than this; STOP as soon as you have no term which is a solution of the homogeneous equation.

Take the guess y_p that you now have, compute y_p' and y_p'', plug these into your non-homogeneous equation, simplify, and collect terms. Compare the expression you get on the left with the function $g(t)$ on the right. The coefficients of the various terms on the left must equal the coefficients of the terms on the right. This gives you a system of equations in the coefficients of your guess. You then solve this system to find y_p.

If you have an equation of the form $ay'' + by' + cy = g_1(t) + g_2(t)$, where $g_1(t)$ and $g_2(t)$ are not the same sort of functions (e.g. $\cos 2t$ and $\cos 3t$ would be an example, but $\cos 2t$ and $\sin 2t$ would not) then you can break the problem into two pieces $ay'' + by' + cy = g_1(t)$ and $ay'' + by' + cy = g_2(t)$, find solutions y_{p_1} and y_{p_2} to each piece, and then let $y_p = y_{p_1} + y_{p_2}$. A similar remark applies to a sum of more than two functions.

2. Variation of parameters.

Consider the equation

$$y'' + p(t)y' + q(t)y = g(t).$$

Suppose we happen to know a fundamental set of solutions $\{y_1, y_2\}$ of the corresponding homogeneous equation. Then we can find a solution y_p of the non-homogeneous equation as follows.

Compute the Wronskian $W = y_1 y_2' - y_2 y_1'$ of y_1 and y_2.

Set $u_1' = -\dfrac{gy_2}{W}$.

Set $u_2' = \dfrac{gy_1}{W}$.

Integrate to get $u_1 = \int u_1' \, dt$.

Integrate to get $u_2 = \int u_2' \, dt$.

You do NOT need a $+C$ in these integrals.

Compute $y_p = u_1 y_1 + u_2 y_2$.

APPLICATIONS

1. Mechanical vibrations.

A mechanical system consisting of a mass m attached to a spring with spring constant k, a dashpot with damping coefficient b, and an external force $F(t)$ is governed by the equation

$$my'' + by' + ky = F(t)$$

where y is the displacement of the mass from its equilibrium position (the point where it is at rest).

You should know that weight and mass are related by $\boxed{W = mg}$. In the metric MKS system distance is measured in meters, mass in kilograms, force and weight in newtons (kg-m/sec^2), and time in seconds, with $g = 9.8$ m/sec^2. In the metric CGS system distance is measured in centimeters (0.01 meter), mass in grams (0.001 kilogram), force and weight in dynes (g-cm/sec^2), and time in seconds, with $g = 980$ cm/sec^2. In the English system distance is measured in feet (1 ft = 12 inches), mass in slugs, force and weight in pounds (1 lb = 1 slug-ft/sec^2), and time in seconds, with $g = 32$ ft/sec^2.

If it takes a force of F_s to stretch a spring by distance L from its unstretched length then $\boxed{F_s = kL}$, where $k > 0$.

If the dashpot exerts a force of F_d for a velocity of v, then $\boxed{F_v = bv}$.

Case 1: Unforced vibrations.

Suppose $F(t) = 0$. Then we have $my'' + by' + ky = 0$. This is a constant coefficient equation which we solve in the standard way.

If $b = 0$, the system is undamped, and the general solution is $u = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$, where $\boxed{\omega_0 = \sqrt{k/m}}$. YOU MUST KNOW THIS FORMULA. We can rewrite this in the form $R \cos(\omega_0 t - \delta)$, where $R = \sqrt{c_1^2 + c_2^2}$, $\cos \delta = c_1/R$, and $\sin \delta = c_2/R$. Thus we get a sinusoidal oscillation with constant amplitude R. We call ω_0 the *natural frequency* of the system. The *period* T is the time between successive oscillations. We have $\omega_0 T = 2\pi$.

If $b \neq 0$, the system is damped. The general solution depends on the relative values of m, b, and k as follows.

If $b^2 - 4mk > 0$, then the characteristic equation has two real roots r_1 and r_2 which are both negative. Then the general solution $c_1 e^{r_1 t} + c_2 e^{r_2 t}$ goes to zero without any oscillations as $t \to \infty$. We say the system is *overdamped*.

If $b^2 - 4mk = 0$, then the characteristic equation has one real root r; it is negative. The general solution $c_1 e^{rt} + c_2 t e^{rt}$ goes to zero without any oscillations as $t \to \infty$. We say the system is *critically damped*.

If $b^2 - 4mk < 0$, then the characteristic equation has two non-real roots $\alpha \pm \beta\, i$; α is negative. The general solution $c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$ goes to zero with oscillations of decreasing amplitude as $t \to \infty$. We say the system is *underdamped*. We call β the *quasi-frequency*. The *quasi-period* T_d is related to it by $\beta T_d = 2\pi$.

YOU MUST BE ABLE TO DETERMINE THE TYPE OF DAMPING A SYSTEM HAS.

<u>Case 2</u>: Forced vibrations.

Now suppose that $F(t)$ is not identically zero. We can apply undetermined coefficients or variation of parameters to find a particular solution y_p to our non-homogeneous equation and then get its general solution as $y = c_1y_1 + c_2y_2 + y_p$, where y_1 and y_2 are fundamental solutions to the homogeneous equation.

The most important case is when $F(t)$ is a sinusoidal function like $F_0 \cos \gamma t$ or $F_0 \sin \gamma t$. We consider for simplicity the cosine case.

In what follows you should not try to memorize the specific formulas for the solutions. In any particular problem you would just solve these equations using the methods developed earlier. If you need any of these formulas they will be given to you. However, you should understand what the general behavior of the solutions will be and when these various types of behavior occur. For example, YOU SHOULD BE ABLE TO DETERMINE WHEN RESONANCE OCCURS.

First consider the undamped case $b = 0$.

Recall that $\omega_0 = \sqrt{k/m}$. If $\omega_0 \neq \gamma$, then our general solution is

$$c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{(k - m\gamma^2)} \cos \gamma t.$$

In general this will have a very complicated graph. In the special case where the initial position and velocity are both zero it can be rewritten as a product of two sine functions, so that it exhibits the phenomenon of *beats*, a rapid oscillation with slowly varying amplitude.

If $\omega_0 = \gamma$, then our general solution is

$$c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t.$$

The third term makes the solution blow up as $t \to \infty$. This is the phenomenon of *resonance*. $\boxed{\text{SO RESONANCE OCCURS WHEN } \omega_0 = \gamma}$.

Next consider the damped case $b \neq 0$. We will still let $\omega_0 = \sqrt{k/m}$

The general solution is $y = y_h + y_p$, where y_h is the solution to the corresponding homogeneous equation. Recall that $y_h \to 0$ as $t \to \infty$. For this reason y_h is called the *transient* solution. So for large values of t we have that y is approximately equal to y_p. So y_p is called the *steady state* solution. It is given by

$$y_p = \frac{F_0}{\sqrt{(k - m\gamma^2)^2 + (b\gamma)^2}} \sin(\gamma t + \theta)$$

where $\tan\theta = (k - m\gamma^2)/(b\gamma)$.

The quantity $M(\gamma) = \dfrac{1}{\sqrt{(k - m\gamma^2)^2 + (b\gamma)^2}}$ is called the *frequency response*; it is a measure of how big the amplitude of y_p is compared to the amplitude F_0 of the forcing function $F(t)$.

Note that if you vary the natural frequency ω_0 or vary the forcing frequency γ, then the amplitude of y_p will vary.

If γ is fixed and ω_0 is varied, then y_p will have a maximum amplitude when $\omega_0 = \gamma$.

If ω_0 is fixed and γ is varied, then y_p will have a maximum amplitude when $\gamma = \sqrt{\omega_0^2 - \dfrac{b^2}{2m^2}}$.

2. Electrical vibrations.

Consider a simple series circuit consisting of an inductor with inductance L measured in henrys, a resistor with resistance R measured in ohms, a capacitor with capacitance C measured in farads, and an external applied voltage $E(t)$ measured in volts. Let Q be the charge on the capacitor measured in coulombs. The current flowing around the circuit will be $I = Q'$ measured in amperes. Then the equation for Q will be

$$LQ'' + RQ' + \frac{1}{C}Q = E(t).$$

This is mathematically the same as the mechanical system described above, with the inductor behaving like a mass, the resistor like a dashpot, the capacitor like a spring (but with constant $1/C$, not C!), and the external voltage like an external force. So all the above discussion goes through in an analogous fashion. You should be able to set up the equation, find the natural frequency ω_0, be able to decide between over, under, and critical damping, find the resonant frequency, etc. One little difference is that sometimes a problem will ask for the current instead of the charge; this is easy to handle since $I = Q'$.