REVIEW FOR EXAM III

This exam covers sections 6.1-6.2 and 7.1-7.5 in the book. As usual this review is basically a checklist. For detailed examples and solutions you should consult your class notes and the book.

The material in Chapter 6 is basically just a generalization of the material in Chapter 4. One has the same basic ideas and techniques, but their execution may be longer and more complicated.

BASIC FACTS ABOUT HIGHER ORDER LINEAR EQUATIONS

1. Existence and uniqueness.

Chapter 6 is concerned with equations of the following form:

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = g(x)$$

An *initial value problem* consists of this equation together with the n requirements $y(x_0) = y_0$, $y'(x_0) = y_0'$, ..., $y^{n-1}(x_0) = y_0^{n-1}$, where x_0, y_0, y_0', ..., y_0^{n-1} are given numbers. A *solution* to the initial value problem is a function $y = \phi(x)$ which satisfies the equation and the initial conditions. As before we can ask whether a solution exists, whether it is unique, and for what values of x it is valid.

Suppose x_0 is contained in an open interval I and that all the $p_i(x)$, and $g(x)$ are continuous for all $x \in I$. Then a solution exists, it is unique, and it is valid for all $x \in I$.

A typical problem concerning existence and uniqueness might give you the initial data and an equation in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

and ask you for the largest interval on which the initial value problem has a unique solution. To find this out divide both sides by $a_n(x)$ to get an equation in the standard form given above. This will tell you what the $p_i(x)$ and $g(x)$ are. Locate the points at which they are discontinuous. These will chop the real number line into pieces which will be open intervals. Find the interval which contains x_0.

2. Homogeneous equations.

(a) Linear independence.

Consider n functions $f_1(x), f_2(x), \ldots, f_n(x)$ defined on an open interval I. (I is an interval of the form $a < x < b$, where you might have $a = -\infty$ and/or $b = \infty$.) They are *linearly dependent* if there are constants $c_1, c_2, \ldots, c_n$ such that:

(i) At least one of these numbers is not zero and

(ii) $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$ for all x in I.

Note that for this definition to really say anything important we must have condition (i); otherwise we could always satisfy (ii) by choosing all c_i to be zero.

The condition that $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ is linearly dependent is equivalent to the condition that at least one of the functions is a linear combination of the other $n-1$ functions.

In order to show that a set of functions is linearly dependent you must know something about the functions and how they are related over the whole interval I. Since the equation is required to hold for *all* x in I, you cannot prove that they are linearly dependent by examining just one point in I. Linear dependence is a property of functions over *intervals* NOT at *points*.

A set of functions is *linearly independent* on I if it is not linearly dependent on I. This is equivalent to the statement that if $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$ for *all* x in I, then all the c_i must be zero.

Unlike the situation with linear dependence it is sometimes possible to prove linear independence by examining a single point in I. Differentiate the equation $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$ to get $c_1 f_1'(x) + c_2 f_2'(x) + \cdots + c_n f_n'(x) = 0$. Continue to differentiate until you get a total of n equations, the last one being $c_1 f_1^{(n-1)}(x) + c_2 f_2^{(n-1)}(x) + \cdots + c_n f_n^{(n-1)}(x) = 0$ Then consider a point x_1 in I. Setting $x = x_1$ we get a system of n equations in the n unknowns c_i having coefficient matrix

$$M = \begin{bmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1'(x_1) & f_2'(x_1) & \cdots & f_n'(x_1) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x_1) & f_2^{(n-1)}(x_1) & \cdots & f_n^{(n-1)}(x_1) \end{bmatrix}.$$

The determinant of this matrix is the *Wronskian* $W[f_1, f_2, \ldots, f_n](x_1)$ of $\{f_1, f_2, \ldots, f_n\}$ at x_1.

If $W[f_1, f_2, \ldots, f_n](x_1) \neq 0$, then the only solution of our system is $c_i = 0$ for all i, and so the functions are linearly independent.

WARNING #1: If $W[f_1, f_2, \ldots, f_n](x_1) = 0$, this does NOT necessarily mean that the functions are linearly dependent. You may simply have made a bad choice for x_1. There may be some other point x_2 in I at which the Wronskian is non-zero, and so they might be lineary independent.

WARNING #2: Even if $W[f_1, f_2, \ldots, f_n](x) = 0$ for *all* x in I, this still does NOT necessarily mean that the functions are linearly dependent. There are examples which illustrate this possibility.

Now, having issued these warnings, you should be aware of the fact that if we happen to know that these functions are solutions of some homogeneous linear differential equation then the situation is dramatically different. In that case W is either never zero or always zero, and being non-zero *is* equivalent to the functions being lineary independent. See below.

Another way to try to show that a set of functions is linearly independent is to choose different points x_1, x_2, …, x_n in I and consider the system of equations $c_1 f(x_j) + c_2 f_2(x_j) + \cdots + c_n f_n(x_j) = 0$ for $j = 1, 2, \ldots, n$. If the only solution is all $c_i = 0$, then the functions are lineary independent. If there is a non-trivial solution, then you don't yet know whether they are linearly dependent or independent. You could try a different set of points to try to show independence or you could try to show dependence by finding a linear relationship among the functions.

(b) Fundamental solutions.

Our equation is *homogeneous* if its right hand side $g(t)$ is zero for all x.

Consider an interval I on which all the $p_i(x)$ are continuous. Let y_1 and y_2 be solutions of the equation. A basic property of solutions of a homogeneous equation is the *principle of superposition*, which says that for any constants c_1 and c_2 the function $c_1 y_1 + c_2 y_2$ is also a solution of the homogeneous equation. From this it follows that if y_1, y_2, …, y_n are solutions, then so is the *linear combination* $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$.

Solutions y_1, y_2, …, y_n of our equation form a *fundamental set of solutions* if *every* solution of the equation on I can be written in the form $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ for some choice of constants c_i.

The basic result about fundamental sets of solutions is that given solutions y_1, y_2, ..., y_n of the homogeneous equation, the following statements are equivalent:

(i) They form a fundamental set of solutions on I.
(ii) They are linearly independent on I.
(iii) Their Wronskian is non-zero for *some* point x_1 in I.
(iv) Their Wronskian is non-zero for *every* point x_1 in I.

One ingredient in this result is *Abel's Theorem*, which says that

$$W[y_1, y_2, \ldots, y_n](x) = c \exp\left(-\int p_1(x)\,dx\right).$$

Since the exponential function is never zero, W will never be zero unless the constant c is zero, in which case W is zero everywhere. So, if you are given n functions whose Wronskian is sometimes zero and sometimes non-zero on I you know that these functions cannot be solutions of an n^{th} order homogeneous linear differential equation on I.

(c) Initial value problems.

If you have a fundamental set $\{y_1, y_2, \ldots, y_n\}$ of solutions of a homogeneous equation and an initial value problem, then differentiating $n-1$ times and plugging in the initial values gives a system of n equations in the n unknowns c_i. Solving for the c_i gives you the solution $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ of the initial value problem.

FINDING SOLUTIONS TO HOMOGENEOUS EQUATIONS

1. Constant coefficient homogeneous equations.

Consider equations of the form

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0y = 0.$$

By plugging $y = e^{rt}$ into this equation and then dividing by e^{rt} we get the *characteristic equation*

$$a_nr^n + a_{n-1}r^{n-1} + \cdots + a_0 = 0.$$

In general, finding the roots of a higher degree polynomial equation is difficult. There are several techniques that one can try. If all the a_i are integers, then any rational root p/q in lowest terms must have p a factor of the constant term a_n and q a factor of a_0. In particular, if $a_0 = 1$, then any rational root must be a factor of the constant term

a_n. You can test whether these factors are roots by using synthetic division to evaluate the polynomial; when a number is a root the synthetic division also factors the polynomial for you.

Another thing to look out for is the binomial theorem, which will give you that the polynomial has the form $(r-a)^n$. Another trick is to notice that sometimes the polynomial is a polynomial in a power of r, such as $r^6 - 5r^3 + 6 = (r^3)^2 - 5(r^3) + 6 = (r^3 - 2)(r^3 - 3)$. Also, when you have a factor of the form $r^k - a$, you can solve the equation $r^k = a$ as follows. Recall Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. Express $r = se^{i\theta}$ and $a = be^{i\phi}$, where s and b are positive real numbers. Then $r^k = a$ implies that $s^k e^{ik\theta} = be^{i\phi}$, which gives $s^k = b$ and $k\theta = \phi + 2\pi m$, where m is an integer. Thus $s = b^{1/k}$ and $\theta = \frac{\phi+2\pi m}{k}$. As m runs through the integers there will be only finitely many different values of $e^{i\theta}$ that appear and so only finitely many values for r. Of course this method is something of a last resort. Look for obvious factorizations and roots first.

The characteristic equation factors into a_0 times a product of two kinds of factors:

Powers of linear polynomials $(r-a)^k$. This will contribute the k functions e^{at}, te^{at}, ..., $t^{k-1}e^{at}$ to our fundamental set of solutions.

Powers of irreducible quadratics $(r^2 + br + c)^m$, where $b^2 - 4c < 0$. The roots are $\alpha \pm \beta i$. This will contribute the $2m$ functions $e^{\alpha t}\cos\beta t$, $e^{\alpha t}\sin\beta t$, $te^{\alpha t}\cos\beta t$, $te^{\alpha t}\sin\beta t$, ..., $t^{m-1}e^{\alpha t}\cos\beta t$, $t^{m-1}e^{\alpha t}\sin\beta t$ to our fundamental set of solutions.

THE LAPLACE TRANSFORM

1. Definitions and properties.

Given a function $f(t)$ which is defined for $t \geq 0$, its *Laplace transform* is the function

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

The function $f(t)$ is *piecewise continuous* if there are numbers t_i chopping the real line into intervals $t_{i-1} < t < t_i$ such that $f(t)$ is continuous on each interval and the limit of $f(t)$ as t approaches an endpoint of the interval from above or below exists. It is of *exponential order* if there are constants $M > 0$, $K > 0$, and a such that for all $t \geq M$ one has $|f(t)| \leq Ke^{at}$. If $f(t)$ is piecewise continuous and of exponential order then its Laplace transform exists.

One computes elementary Laplace transforms using calculus. Here are some basic ones. On the exam you will be given a sheet with these transforms as well as some some other Laplace transform formulas mentioned below. To see precisely what information about Laplace transforms you will be given view the table posted on Canvas in the Course Information Section of Content.

$f(t)$	$F(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$

Note that the Laplace transforms of e^{at}, $e^{at}\cos bt$, $e^{at}\sin bt$, and $t^n e^{at}$ are obtained from those of 1, $\cos bt$, $\sin bt$, and t^n, respectively, by replacing s by $s-a$. This is due to the *shift property*

$$\mathcal{L}\{e^{at}g(t)\} = G(s-a),$$

where $G(s) = \mathcal{L}\{g(t)\}$.

Another important property of the Laplace transform is *linearity*:

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

The property of the Laplace transform which makes it useful for differential equations is the *derivative property*

$$\mathcal{L}\{g^{(m)}(t)\} = s^m G(s) - s^{m-1}g(0) - s^{m-2}g'(0) - \cdots - g^{m-1}(0),$$

where $G(s) = \mathcal{L}\{g(t)\}$. So we have $\mathcal{L}\{g'(t)\} = sG(s) - g(0)$, $\mathcal{L}\{g''(t)\} = s^2G(s) - sg'(0) - g'(0)$, $\mathcal{L}\{g'''(t)\} = s^3G(s) - s^2g(0) - sg'(0) - g''(0)$, and so on.

2. Initial value problems.

Now, given an initial value problem, let $y(t)$ be our unknown solution and $Y(s) = \mathcal{L}\{y(t)\}$ its Laplace transform. We solve the problem in two steps. First we find $Y(s)$. Then we *invert* $Y(s)$ to find $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

To find $Y(s)$ we take the Laplace transform of both sides of our equation. Use linearity to break everything up into individual terms. Then use the derivative property to express the transforms of the derivatives of $y(t)$ into expressions involving s, $Y(s)$, and the initial values. Plug in the initial values. Then solve the resulting algebraic equation for $Y(s)$. This will usually consist of one or more rational functions (quotients of polynomials).

To invert $Y(s)$ to find $y(t)$ we must write $Y(s)$ as a linear combination of the transforms $F(s)$ given in our table. Then we can simply read off $y(t)$ by replacing each $F(s)$ by the corresponding $f(t)$.

Note that each $F(s)$ in the table given here is a *partial fraction*, that is, it has the form $\dfrac{A}{(s-a)^k}$ or the form $\dfrac{Bs+C}{(s^2+ps+q)^k}$ where s^2+ps+q is an irreducible quadratic ($p^2 - 4q < 0$). This would be a good time to review partial fractions from your calculus course, where they were used to integrate rational functions. We give a brief summary here.

Suppose $Y(s)$ is a single rational function. Our goal is to express it as a sum of partial fractions. The denominator of $Y(s)$ factors into a product of linear functions $(s-a)^m$ and irreducible quadratics $(s^2+ps+q)^m$.

Each $(s-a)^m$ factor contributes

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_m}{(s-a)^m}$$

to the sum.

Each $(s^2+ps+q)^m$ factor contributes

$$\frac{B_1 s + C_1}{s^2+ps+q} + \frac{B_2 s + C_2}{(s^2+ps+q)^2} + \cdots + \frac{B_m s + C_m}{(s^2+ps+q)^m}$$

to the sum.

You MUST have all these terms present to ensure a solution!

Now set $Y(s)$ equal to the sum of all the relevant partial fractions. Add all the partial fractions to get a rational function with a denominator which is equal to the denominator of $Y(s)$. Then the numerators will be equal. There are now two ways to proceed. One way is to expand out the numerator and collect terms in powers of s, then set the coefficients of like powers equal. The other (the *Heaviside trick*) is to plug in various values for s in both numerators, choosing especially those values which make terms vanish. Either way you get a system of linear equations in the A_k, B_k, and C_k, which you then solve.

So now we have a sum of partial fractions. What do we do with it? Any partial fraction of the form $\dfrac{A}{(s-a)^k}$ can be expressed as a constant times one of the $F(s)$ in our table and so is easy to invert.

So suppose we have something of the form $\dfrac{Bs+C}{(s^2+ps+q)^k}$ where s^2+ps+q is an irreducible quadratic. By completing the square we can write $s^2+ps+q=(s-a)^2+b^2$. This suggests that the transforms of $e^{at}\cos bt$ and $e^{at}\sin bt$ are involved. Suppose $k=1$. Then we want to write

$$\frac{Bs+C}{(s-a)^2+b^2}=\alpha\left(\frac{s-a}{(s-a)^2+b^2}\right)+\beta\left(\frac{b}{(s-a)^2+b^2}\right).$$

Looking at the numerators, we have $B=\alpha$ and $C=-\alpha a+\beta b$. We solve this for α and β and thus get that the inverse transform is

$$\alpha e^{at}\cos bt+\beta e^{at}\sin bt.$$

The situation for $k>1$ can be handled using the method of convolutions which is described section 7.8 (but will not be covered in this course).

3. The t-multiplication property

   This property says that if $\mathcal{L}\{f(t)\}=F(s)$, then

   $$\mathcal{L}\{t\,f(t)\}=-F'(s)$$

   For example, let $g(t)=t\,\sin(bt)$. Then $f(t)=\sin(bt)$, and $F(s)=b/(s^2+b^2)$. We have $F'(s)=-2bs/(s^2+b^2)^2$, and so we have that $G(s)=2bs/(s^2+b^2)^2$.

   Note that this allows us to invert the partial fraction $s/(s^2+b^2)^2$ to get $t\,\sin(bt)/(2b)$.