Weighted Adaptive Multiple Decision Functions for False Discovery Rate Control

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Outline

Motivation

Introduction: Motivation of Multiple Testing and FDR Research Areas

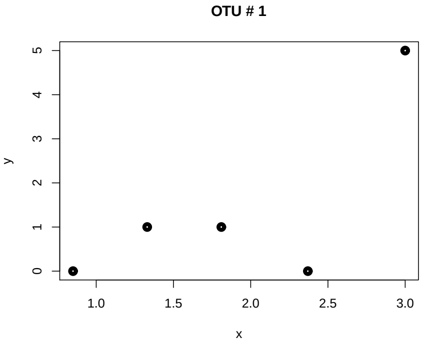
Data: Anderson and Habiger (2012)

Biology Theory: Ecosystem of micro-organisms (OTUs) near the roots of wheat (stomache of wheat plant).

Question: Which species (OTUs) are associated with productivity?

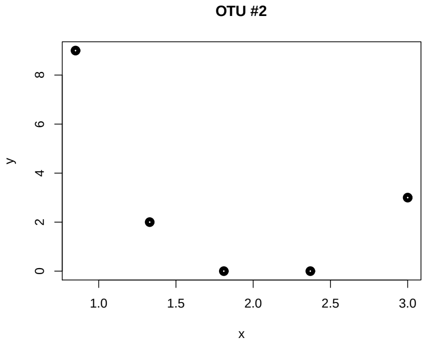
OTU #	Productivity/Shoot Biomass (g)				
	0.85	1.33	1.81	2.37	3.00
1	0	1	1	0	5
2	9	2	0	0	3
⋮	⋮	⋮	⋮	⋮	⋮
M = 778	16	10	29	18	13

Goal 1



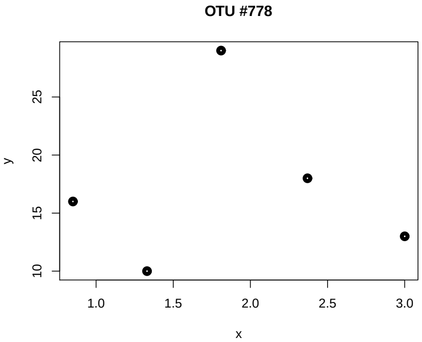
Y (prevelence) vs. X (productivity)?

Goal 2



Y (prevelence) vs. X (productivity)?

Goal 778



Y (prevelence) vs. X (productivity)?

Model

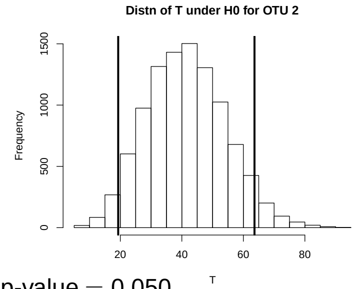
- Data: Y_{mj} prevelence of mth OTU in jth productivity group
- Model:

$$Y_{mj} \sim Poisson(e^{\beta_{0m}+\beta_{1m}x_j})$$

- Hypotheses: $H_{0m} : \beta_{1m} = 0$ vs. $H_{1m} : \beta_{1m} \neq 0$
- Sufficient Stat: $T_m = \sum_{j=1}^{5} x_j Y_{mj}$
- Ancillary Stat: $Y_{m\cdot} = \sum_{j=1}^{5} Y_{mj}$

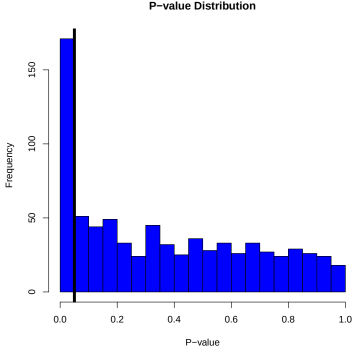
A single test

- $Y_{m1}, Y_{m2}, ..., Y_{m5} | Y_{m\cdot} \sim Multi(Y_{m\cdot}, 1/5)$ under H_{m0}



- $T_2 = 19.31 \rightarrow$ p-value $= 0.050$.
- Reject $H_{02} : \beta_{12} = 0$?

Multiple Tests



Using p-value cutoff $\alpha = 0.05$ ...

- **171 Discoveries** ("productivity associated OTU's")
- **778 $\times$ 0.05 = 49 False Discoveries**

Main Error Rates

1. **Family-Wise Error Rate:**

$$FWER = \Pr(\#\text{False Discoveries} \geq 1)$$

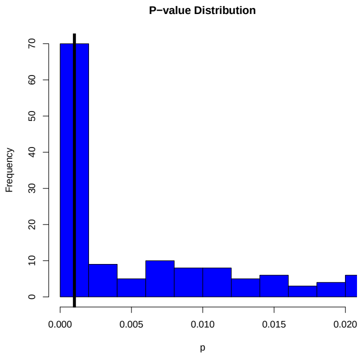
   - Bonferroni: **p-value** $\overset{?}{\leq} \frac{0.05}{778}$ $\rightarrow$ **54 discoveries**

2. **False Discovery Rate:**

$$FDR = E\left[\frac{\#\text{ False Discoveries}}{\#\text{ Discoveries}}\right]$$

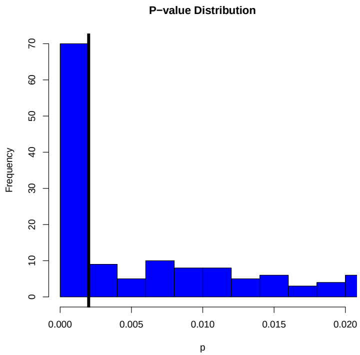
   - BH procedure: **p-value** $\overset{?}{\leq} 0.005 \rightarrow$ **82 discoveries**
     - Benjamini and Hochberg (1995)

Illustration of BH procedure



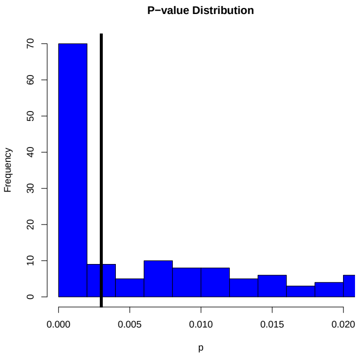
$$\widehat{FDR}(\color{red}{0.001}) = \frac{778 \times \color{red}{0.001}}{54} = 0.014$$

Illustration of BH procedure



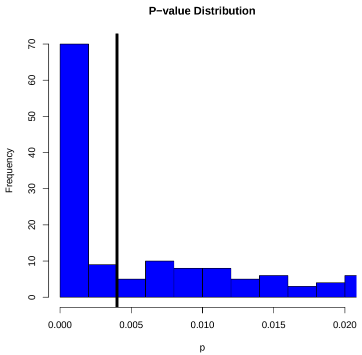
$$\widehat{FDR}(0.002) = \frac{778 \times 0.002}{64} = 0.024$$

Illustration of BH procedure



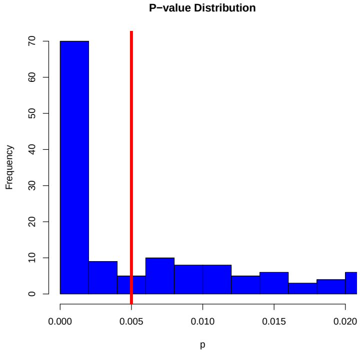
$$\widehat{FDR}(0.003) = \frac{778 \times 0.003}{70} = 0.033$$

Illustration of BH procedure



$$\widehat{FDR}(0.004) = \frac{778 \times 0.004}{74} = 0.042$$

Illustration of BH procedure



$$\widehat{FDR}(\mathbf{0.005}) = \frac{778 \times \mathbf{0.005}}{82} \approx \mathbf{0.05}$$

Properties of BH Procedure

Theorem (Benjamini and Hochberg; 1995)

If P-values from true null hypotheses are

1. *independent*
2. *uniformly distributed*

then the BH procedure has FDR $\leq a_0 \alpha \leq \alpha$

- $a_0 =$ proportion of true null hypotheses

FDR Areas

FDR Research Areas

Uniform Distribution

P-values not uniformly distributed under null if

1. Model misspecified
   - Efron (2001, 2007); Pollard and van der Laan(2004)
   - Habiger and Peña (2011)
2. Data/test statistic discrete
   - Lancaster (1961); Pratt(1961)
   - Geyer and Meeden (2005); Kulinskaya and Lewin (2009)
   - Gutman and Hochberg (2007)
   - Habiger (2013)

Dependence

P-values may not be independent under null

1. Positive Dependence
   - Benjamini and Yekuteuli (2001)
   - Guo, Li, and Sarkar (2013), Guo and Sarkar(2013)
2. Weak Dependence
   - Storey (2001), Genovese and Wasserman (2002), Storey, Taylor and Siegmund (2004)
3. Arbitrary/Strong Dependence
   - Benjaminin and Yekuteuli (2001)
   - Efron (2012); Desai and Storey (2012); Fan, Han and Gu (2012)

Exhausting the α

BH: $FDR = a_0\alpha < \alpha$

1. Exact adaptive FDR control via $\hat{a}_0$
   - Storey, Taylor and Siegmund (2004); Benjamini, Krieger and Yekuteli (2006); Gavrilov, Benjamini and Sarkar (2009); Liang and Nettleton (2012)
2. Consistent estimation of a_0
   - Sun and Cai (2007); Jin and Cai (2007); . . MANY more
3. Asymptotically Optimal Rejection Curve
   - Finner et. al (2009)

Heterogeneity

Data not homogeneous - Example ancillary statistics

1. weighted BH type procedures
   - Genovese and Wasserman (2006); Roeder and Wasserman (2009); Roquain and van de Wiel (2009); Peña, Habiger and Wu (2011)
2. focus on group/cluster structure
   - Sun and Cai (2009); Hu, Zhao and Zhou (2010)

P-value efficiency

- local FDR for $z-$scores: $lfdr(z) = \frac{pf_0(z)}{f(z)}$
    - Efron et. al (2001);Efron(2010); Sun and Cai (2007); Jin and Cai (2007)
    - Rubin, Dudoit and van der Laan (2006); Habiger and Peña (2013)
- P-value approach = Z-value approach if p-value appropriately defined
    - Habiger (2012)

Idea of this Paper

Weak Dependence + Weighted + Adaptive

Finite FDR Control

Preliminaries and Finite FDR Control

Data and Hypotheses

- Data: $\boldsymbol{Z} = (Z_m, m \in \mathcal{M}) \sim F$ for $\mathcal{M} = \{1, 2, ..., M\}$
  - Example: $\boldsymbol{Z} \sim MVN(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
- Null hypotheses: $\boldsymbol{H} = (H_m, m \in \mathcal{M})$
  - Example: $H_m : \mu_m = 0$
- True nulls: $\mathcal{M}_0 = \{m \in \mathcal{M} : H_m \text{ true } \} \subseteq \mathcal{M}$
- Number true nulls: $M_0 = |\mathcal{M}_0|$

Decision functions

Decision fxn: $\delta_m(Z_m; t_m) \in \{0, 1\}$ - t_m a "threshold" (or size or index)

- Example 1: $\delta_m(Z_m; t_m) = I(Z_m \geq \bar{\Phi}^{-1}(t_m))$
- Example 2: p-value $\rightarrow \delta_m(P_m(Z_m); t_m) = I(P_m(Z_m) \leq t_m)$
- Assumptions
  1. $E[\delta_m(Z_m; t_m)] = t_m$ under H_m
  2. $t_m \mapsto E[\delta_m(Z_m; t_m)]$ continuous and strictly increasing
- Shorthand: $\delta_m(t_m)$ a Bernoulli process

Multiple decision fxn: $\boldsymbol{\delta}(\boldsymbol{t}) = [\delta_m(t_m), m \in \mathcal{M}]$

- $\boldsymbol{t} = (t_m, m \in \mathcal{M})$ - "threshold vector"

False Discovery Proportion/Rate

False Discovery Proportion (FDP):

$$FDP(\mathbf{t}) = \frac{V(\mathbf{t})}{R(\mathbf{t}) \vee 1}$$

False Discovery Rate (FDR):

$$FDR(\mathbf{t}) = E\left[FDP(\mathbf{t})\right]$$

- $V(\mathbf{t}) = \sum_{m \in \mathcal{M}_0} \delta_m(t_m)$ number false discoveries
- $R(\mathbf{t}) = \sum_{m \in \mathcal{M}} \delta_m(t_m)$ number discoveries

Weights and a single threshold

Goal: Choose $\boldsymbol{t}$ large as possible s.t. $FDR(\boldsymbol{t}) \leq \alpha$

$$\Downarrow$$

$$t_m = \bar{t}\left(\frac{t_m}{\bar{t}}\right) \equiv t w_m$$

$\swarrow$ $\searrow$

choose t choose $\boldsymbol{w}$

Procedure: Fixed weights

1. Choose threshold

$$t_\alpha^\lambda = \sup\{0 \le t \le \lambda : \widehat{FDR}^\lambda(t\boldsymbol{w}) \le \alpha\}$$

where

$$\widehat{FDR}^\lambda(t\boldsymbol{w}) = \frac{\hat{M}_0(\lambda\boldsymbol{w})t}{R(t\boldsymbol{w}) \vee 1}$$

2. Compute $\delta(t_\alpha^\lambda \boldsymbol{w})$
   - Weighted adaptive multiple decision function (WAMDF)"

Details

$$\hat{M}_0(\lambda \boldsymbol{w}) = \frac{M - R(\lambda \boldsymbol{w}) + 1}{1 - \lambda}$$

- Weighted version of Storey et. al (2004) estimator
- Intuition for $\boldsymbol{w} = \mathbf{1}$ and $\lambda = 1/2$:
  - If all alternative p-values near 0 then

$$\hat{M}_0 = \{\#p - values > 1/2\} \times 2$$

- Should have $\lambda w_m < 1$

FDR bound

Lemma (FDR bound)

For $M_0 \geq 1$ and under

(A1) $(Z_m, m \in \mathcal{M}_0)$ *independent collection*

(A2) $\lambda w_m < 1$ *for every m,*

$$FDR(t_\alpha^\lambda \boldsymbol{w}) \leq \alpha \bar{w}_0 \frac{1-\lambda}{1-\lambda \bar{w}_0}$$

- $\bar{w}_0 = M_0^{-1} \sum_{m \in \mathcal{M}_0} w_m$ average null weight
- Corollary: $\boldsymbol{w} = \mathbf{1} \Rightarrow \bar{w}_0 = 1 \Rightarrow$ Theorem 3 - Storey et al (2004)

FDR Control

Problem: $\bar{w}_0$ unobservable!

Theorem (FDR control)

Let $w_{(M)} = \max\{w_m, m \in \mathcal{M}\}$ *and define*

$$\alpha^* = \alpha \frac{1}{w_{(M)}} \frac{1 - \lambda w_{(M)}}{1 - \lambda}.$$

Then under (A1) and (A2), $FDR(t^{\lambda}_{\alpha^*}\mathbf{w}) \leq \alpha$.

- Generally have $\bar{w}_0 \leq 1 \ldots$

Asymptotic results

Asymptotic FDP Control

- (Weighted) adaptive vs. unadaptive vs. Oracle
- "α-exhaustion"

Remark: Notation - $\boldsymbol{w}_{0,M}, \widehat{FDR}_M, \ldots$

Assumptions

- Weak Dependence: for $0 \leq t \leq u$,

  (A3) $R(t\mathbf{w}_M)/M \to G(t)$ a. s.

  (A4) $V(t\mathbf{w}_M)/M \to a_0\mu_0 t$ a. s.

  - $\bar{w}_{0,M} \to \mu_0$ asymptotic mean null weight
  - $\frac{M_0}{M} \to a_0$ asymptotic proportion true nulls
  - $u = \sup\{t : tw_m \leq 1\}$

- FDR controllable:

  (A5) $t/G(t)$ is strictly increasing and continuous with

$$\lim_{t\downarrow 0} \frac{t}{G(t)} = 0 \quad \text{and} \quad \lim_{t\uparrow u} \frac{t}{G(t)} = \frac{u}{G(u)} \leq 1,$$

- Similar to Genovese and Wasserman (2006), Storey et. al(2004), ...

FDR Estimators

Estimators + limits

Unadaptive $\widehat{FDR}^{0}_{M}(t\boldsymbol{w}_M) = \frac{Mt}{R(t\boldsymbol{w}_M)} \to \frac{t}{G(t)} \equiv \widehat{FDR}^{0}_{\infty}(t)$

Adaptive $\widehat{FDR}^{\lambda}_{M}(t\boldsymbol{w}_M) = \frac{\hat{M}_0(\lambda\boldsymbol{w}_M)t}{R(t\boldsymbol{w}_M)} \to \frac{1-G(\lambda)}{1-\lambda}\frac{t}{G(t)} \equiv \widehat{FDR}^{\lambda}_{\infty}(t)$

Oracle $FDP_M(t\boldsymbol{w}_M) = \frac{V(t\boldsymbol{w}_M)}{R(t\boldsymbol{w}_M)} \to \frac{a_0\mu_0 t}{G(t)} \equiv FDP_\infty(t)$

Lemma (Glivenko-Cantelli)

Under (A2) - (A5), convergence is uniform for each "estimator" a.s., i.e.

$$\sup_{0\le t\le u} \left|\widehat{FDR}_M(t\boldsymbol{w}) - \widehat{FDR}_\infty(t)\right| \to 0 \qquad \textit{almost surely}$$

Thresholds

Example: $t^{\lambda}_{\alpha,M} = \sup\{0 \leq t \leq u : \widehat{FDR}^{\lambda}_{M}(t\mathbf{w}) \leq \alpha\}$

	Threshold	
	Finite (M) (random)	Asymptotic (∞) (not random)
Unadaptive - $\widehat{FDR}^{0}$	$t^{0}_{\alpha,M}$	$t^{0}_{\alpha,\infty}$
Adaptive - $\widehat{FDR}^{\lambda}$	$t^{\lambda}_{\alpha,M}$	$t^{\lambda}_{\alpha,\infty}$
Oracle FDP	$t_{\alpha,M}$	$t_{\alpha,\infty}$

Threshold comparison

Theorem (Thresholds converge)

Consider

$$\lim_{M\to\infty} t^0_{\alpha,M} = t^0_{\alpha,\infty} \le \lim_{M\to\infty} t^\lambda_{\alpha,M} = t^\lambda_{\alpha,\infty} \le \lim_{M\to\infty} t_{\alpha,M} = t_{\alpha,\infty}$$

Under (A2) - (A5),

- *all equalities + first inequality satisfied a.s.*
- *last inequality satisfied a.s. if* $\mu_0 \le 1$.

- Thresholds constant
- Adaptive threshold $>$ unadaptive threshold

FDP comparison

Theorem (FDPs converge)

$$\lim_{M\to\infty} FDP_M(t^0_{\alpha,M}\mathbf{w}_M) \leq \lim_{M\to\infty} FDP_M(t^\lambda_{\alpha,M}\mathbf{w}_M) \leq \lim_{M\to\infty} FDP_M(t_{\alpha,M}\mathbf{w}_M) = \alpha$$

Under (A2) - (A5),

- *first inequality and last equality satisfied a.s.*
- *last inequality satisfied a.s. if* $\mu_0 \leq 1$

- Important: FDP $\neq$ FDR - Error control on 1 replication
- Q: Does $FDP_M(t^\lambda_{\alpha,M}\mathbf{w}_M) \to \alpha$?

α-exhaustion idea

"Good procedures should be *α-exhaustive*" - Have $FDR_M \to \alpha$ under some "least favorable distribution"

- Finner et. al (2009); Roquain and Villers (2011) - AORC:

$$p_{(k)} \leq \alpha \frac{k}{M - k(1-\alpha)}$$

- Least favorable dist under i.i.d. is "Dirac-Uniform (DU)"

$$E[\delta_m(t)] = \begin{cases} t & m \in \mathcal{M}_0 \\ 1 & m \in \mathcal{M}_1 \end{cases}$$

- Step-up procedures fail

α-exhaustion

Theorem (α-exhaustion)

If $\mu_0 = 1$, then under weak dependence (A3 - A4) and a DU distribution, $FDP_M(t^{\lambda}_{\alpha,M}\mathbf{w}_M) \to \alpha$ almost surely, i.e. $\delta(t^{\lambda}_{\alpha,M}\mathbf{w}_M)$ is α-exhaustive.

Corollary (Adaptive BH is α-exhaustive)

The (unweighted) adaptive BH procedure step up procedure - Storey et. al (2004)- is "α-exhaustive" under weak dependence (A3 - A4).

- Extends Finner et. al (2009) theory to 1) weighted 2) adaptive 3) step-up procedure under 4) weak dependence

Question

Questions

1. α-exhaustion $\Rightarrow$ “optimal”
2. How to choose $\boldsymbol{w}$

Optimal Weights

Optimal Weights

- Mixture model
- Optimal weights for fixed t
- Optimal weights for approximation of $t_{\alpha,\infty}^{\lambda}$

Mixture Model

Model (Random effects model)

Let $(Z_m, \theta_m, p_m, \gamma_m)$, $m \in \mathcal{M}$ *be i.i.d. random vectors with*

$$F(z_m|\theta_m, \gamma_m) = (1-\theta_m)F_0(z_m) + \theta_m F_1(z_m|\gamma_m)$$

and

$$F(z_m|p_m, \gamma_m) = (1-p_m)F_0(z_m) + p_m F_1(z_m|\gamma_m).$$

- $\theta_m \in \{0,1\}$ *with* $p_m = \Pr(\theta_m = 1)$ *ind. of* γ_m
- $E[p_m] = 1 - a_0$ *for* $0 < a_0 < 1$.

- $F_0(\cdot)$ "null distribution" and $F_1(\cdot|\gamma_m)$ "alternative distribution"
- Heterogeneity: effect size γ_m and prior probability p_m

Power function

For $t_m = tw_m$,

$$\pi_{\gamma_m}(t_m) = E[\delta_m(Z_m; t_m)|\theta_m = 1, \gamma_m]$$

is the power function

(A6) $t_m \mapsto \pi_{\gamma_m}(t_m)$ is concave and twice differentiable for $t_m \in (0, 1)$ with $\lim_{t_m \uparrow 1} \pi'_{\gamma_m}(t_m) = 0$ and $\lim_{t_m \downarrow 0} \pi'_{\gamma_m}(t_m) = \infty$ a.s.

- Similar to Genovese and Wasserman (2006); Peña, Habiger, Wu (2011), ...

Optimality goal

1. Assume $\bar{t} = t$ fixed $\Leftrightarrow$ (ex. Bonferroni $t = \alpha/M$)
2. Goal: Maximize ETP / power

$$\pi(\boldsymbol{t}, \boldsymbol{p}, \gamma) \equiv E\left[\sum_{m \in \mathcal{M}} \theta_m \delta_m(t_m) \Big| \gamma, \boldsymbol{p}\right] = \sum_{m \in \mathcal{M}} p_m \pi_{\gamma_m}(t_m),$$

subject to $\bar{t} = t$.

3. Can always recover weights

$$t_m = t w_m \Rightarrow w_m = \frac{t_m}{t}$$

Optimal fixed-t threshold/weights

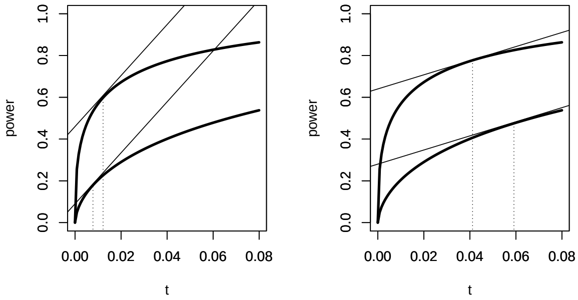
Theorem (Optimal Fixed-t threshold)

Under (A6) and the random effects model, for any fixed $t \in (0,1)$, the maximum of $\pi(\mathbf{t}, \mathbf{p}, \gamma)$ with respect to $\mathbf{t}$ subject to constraint $\bar{t} = t$ exists, is unique, and satisfies (a.s.)

$$\pi'_{\gamma_m}(t_m) = k/p_m$$

1. For any k can compute $t_m(k/p_m, \gamma_m)$
2. Find the k^* satisfying constraint
3. Compute optimal fixed-t weights $w_m^* = \frac{t_m(k^*/p_m, \gamma_m)}{t}$

Illustration: $p_1 = p_2 = 1/2$



- $t = 0.01, 0.05 \Rightarrow k^* = 6.1, 1.7$

Approximation Idea

Problem: $t^{\lambda}_{\alpha,M}$ not fixed so previous theorem not applicable

Solution: Recall $t^{\lambda}_{\alpha,M} \to t^{\lambda}_{\alpha,\infty}$

- Can approximate $t^{\lambda}_{\alpha,\infty}$ "well" using $\boldsymbol{p}$ and γ (details omitted)

Assessment

Assessment

Notation

- Asymptotically optimal fixed-t weights $\boldsymbol{w}_M^*$ (using $t_{\alpha,\infty}^{\lambda}$)
- Approximately optimal fixed-t weights $\hat{\boldsymbol{w}}_M$ (using approximation $t_{\alpha,\infty}^{\lambda}$)
- Pertubed fixed-t weights $\tilde{\boldsymbol{w}}_M$ ($\tilde{w}_{m,M} = U_m \hat{w}_{m,M}$)

FDP control

Theorem (FDP control)

Under the Random Effects Model and (A6), conditions (A2) - (A5) are satisfied and $\mu_0 \leq 1$. *Hence, almost surely*

$$\lim_{M\to\infty} FDP_M(t^0_{\alpha,M}\tilde{\mathbf{w}}_M) \leq \lim_{M\to\infty} FDP_M(t^\lambda_{\alpha,M}\tilde{\mathbf{w}}_M) \leq \alpha$$

For large M

1. adaptive weighted procedure DOMINATES unadpative weighted procedure and is **valid** even if weights are misspecified

α-exhaustion

Theorem (α-exhaustion)

Under model 1 and (A6), $\delta(t_{\alpha,M}\tilde{\boldsymbol{w}}_M)$ is α-exhaustive if $p_1 = p_2 = ... = p_M = p$

1. Many optimal weighting schemes can be motivated using random effects model with $p_1 = p_2 = ... = p_M$
   - Spjotvoll (1972), Genovese and Wasserman (2006); Storey(2007); Peña, Habiger, Wu (2011)
2. If used with $t^{\lambda}_{\alpha,M} \rightarrow$ Optimally weighted + α-exhaustive

Optimal Weights

Theorem (Asymptotically optimal weights)

Under Model 1 and (A6), $\delta_m(t^{\lambda}_{\alpha,M}\hat{w}_m) \to \delta_m(t^{\lambda}_{\alpha,\infty}w^*_m)$ *almost surely for every m.*

- Approximately optimal weights are asymptotically optimal

Simulation setup

Data: $Z_m \sim N(\theta_m \gamma_m, 1)$
Decision: $\delta_m(Z_m; t_m) = I(Z_m \geq \bar{\Phi}^{-1}(t_m))$
Effect sizes: $\gamma_m \overset{i.i.d.}{\sim} Un(1,1), Un(1,3)$, or $Un(1,5)$
Procedures:

- UU $(\mathbf{1}, t^0_{\alpha,M})$ unweighted + unadaptive - Benjamini and Hocbger (1995)
- WU $(\boldsymbol{w}^*_M, t^0_{\alpha,M})$ weighted + unadaptive - Genovese and Wasserman (2006); Peña, Habiger, Wu (2011)
- UA $(\mathbf{1}, t^\lambda_{\alpha,M})$ = unweighted + adaptive - Storey et. al (2004)
- WA $(\boldsymbol{w}^*_M, t^\lambda_{\alpha,M})$ weighted + adaptive - Habiger (201?)

Some Heterogeneity

Simulation 1

	$\gamma_m = 1$	$\gamma_m \overset{i.i.d.}{\sim} Un(1,3)$	$\gamma_m \overset{i.i.d.}{\sim} Un(1,5)$
UU	0.007(0.025)	0.390(0.024)	0.707(0.025)
WU	0.007(0.025)	0.395(0.025)	0.729(0.025)
UA	0.009(0.030)	0.454(0.034)	0.753(**0.039**)
WA	0.009(0.030)	0.457(0.035)	0.778(**0.039**)

Average Power (FDR) for $p_m = 1/2$

- WA - Optimally Weighted + α-exhaustive
- UA - α-exhaustive

More Heterogeneity

Simulation 2

	$\gamma_m = 1$	$\gamma_m \overset{i.i.d.}{\sim} Un(1,3)$	$\gamma_m \overset{i.i.d.}{\sim} Un(1,5)$
UU	0.007(0.025)	0.390(0.025)	0.711(0.025)
WU	0.012(0.012)	0.431(0.015)	0.755(0.016)
UA	0.009(**0.026**)	0.457(**0.035**)	0.757(**0.039**)
WA	0.017(0.015)	0.504(0.021)	0.807(**0.026**)

Average Power (FDR) for $p_m \overset{i.i.d}{\sim} Un(0,1)$

- WA Optimally weighted
- UA - α-exhaustive

Pertubed weights

Simulation 3

	$\gamma_m = 1$	$\gamma_m \overset{i.i.d.}{\sim} Un(1,3)$	$\gamma_m \overset{i.i.d.}{\sim} Un(1,5)$
UU	0.006(0.025)	0.391(0.026)	0.709(0.025)
WU	0.012(0.013)	0.394(0.016)	0.724(0.016)
UA	0.008(0.028)	0.458(0.036)	0.756(0.039)
WA	0.019(0.015)	0.457(0.022)	0.773(0.026)

Average Power (FDR) for $p_m \overset{i.i.d}{\sim} Un(0,1)$ and perturbed weights

- UA - α-exhaustive

Concluding Remarks

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Summary of results

- Optimality of weighted adaptive procedure
  - Heterogeneity and adaptive vs. Heterogeneity or adaptive
- Robustness of weighted adaptive procedure
  - FDP (not FDR) control under weak dependence and noisy weights
  - Good power under noisy weights

Near future work

OTU#	Sufficient Stat	Ancillary Stat	weights
1	18.14	7	?
2	19.13	14	?
⋮	⋮	⋮	⋮
M = 778	161.05	87	?

1. How can we estimate γ_ms / p_ms ?
2. Different optimality goal: maximize $\sum_m \sum_j (\hat{y}_{mj} - \bar{y}_{m\cdot})^2$ s.t. 5% of "declared productivity-associated variability is falsely declared variability"

Near future work continued

1. Choice of λ?
   - Dynamic - Liang and Nettleton (2012)
   - Variance vs. bias vs. power
2. Estimation of μ_0?

General Future work

Optimally weighted

+

α - exhaustive

+

strong dependence

+

discrete data

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